Coating Flow on to Rods and Wires

S. D. R. Wilson
Department of Mathematics
University of Manchester
Manchester M13 9PL, England

It is of some interest to predict the amount of fluid entrained when a rod or wire is withdrawn vertically from a bath of liquid. Both experimental and theoretical results, for example, were presented by White and Tallmadge (1967); however, although the agreement between the two is quite satisfactory, the theory is nevertheless mistaken and unnecessarily complicated.

The similar, but distinctly easier, problem of entrainment on a vertically withdrawn flat plate was considered by Wilson (1982). There it was shown that difficulties encountered in previous theories were caused by irrational methods of approximation, and a correct analysis was given which is asymptotically valid in the limit of small capillary number $Ca = \mu U/\gamma$. If Ca is not small the viscous stresses normal to the free surface cannot be neglected and there appears to be no possibility of a theoretical treatment. Both the earlier theories and the new theory (which does not differ numerically from them to any great extent) agreed with experiment for values of Ca of order unity, but this appears to be no more than a fortunate chance.

The purpose of the present paper is to present a similar theory for the coating of cylinders and wires. This is complicated by the presence of an additional parameter, the wire radius, but is generally similar. As before it is asymptotically valid for small Ca. The formula for the coating thickness contains an expression for the height to which the hydrostatic meniscus would rise on the (stationary) cylinder, for which no analytical expression of general validity can be found. However, an accurate numerical solution is available (Hildebrand et al, 1970); if this is used, the present formula holds good for almost all cylinder radii.

Analysis

To keep the exposition simple and readable the derivation will be supported by plausible arguments rather than the immense formal calculations which would be necessary for full rigor. Much of the detail has been presented elsewhere in connection with the plane drag-out problem (Wilson, 1982).

Cylindrical polar coordinates are chosen as indicated in Figure 1, with Oz directed downwards. The flow is described by the equations of lubrication theory, which are

$$0 = g - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right], \tag{1}$$

$$p = \gamma\kappa = \gamma(\kappa_R - \kappa_A), \tag{2}$$

where

$$\kappa_A = \frac{d^2 s}{dz^2}\left\{1 + \left(\frac{ds}{dz}\right)^2\right\}^{-3/2}, \quad \kappa_R = s^{-1}\left\{1 + \left(\frac{ds}{dz}\right)^2\right\}^{-1/2}, \tag{3}$$

and $s = R + h$. The procedure is to integrate Eq. 1 for the velocity w, using the boundary conditions.

$$\left.\begin{aligned} w &= -U \text{ on } r = R \\ \frac{\partial w}{\partial r} &= 0 \text{ on } r = R + h \end{aligned}\right\} \tag{4}$$

and then carry out a further integration with respect to r to obtain flux, Q. When h is small compared to R, this comes to

$$-URh + \frac{1}{3}\frac{Rh^3}{\mu}\left(\rho g + \gamma\frac{d\kappa}{dz}\right) = Q/2\pi = -URH + \frac{1}{3}\frac{RH^3\rho g}{\mu}, \tag{5}$$

where H is film thickness at minus infinity. It is H that we wish to find, of course. [This equation corresponds to Eq. of Wilson (1982).]

Equation 5 is an ordinary differential equation for the film thickness $h(z)$ and the crucial step in the solution is the correct matching to the horizontal free surface of the liquid bath. Note that for small Ca, Eq. 5 remains valid in the meniscus even though the free surface turns through a right angle. This is because the terms which lubrication theory fails to estimate correctly in the meniscus are neglected there anyway, on order-of-magnitude grounds. The main balance is between surface tension and gravity forces which are correctly estimated. In this respect, the present problem may be distinguished from the flow at the exit from a nip, considered by (for example) Ruschak (1982) and Williamson (1972); in this case, gravity is ignored and the meniscus represents a balance between surface tension and viscous forces.

The matching is done best using suitable dimensionless variables, but it is not clear what these are. We can start by intro-

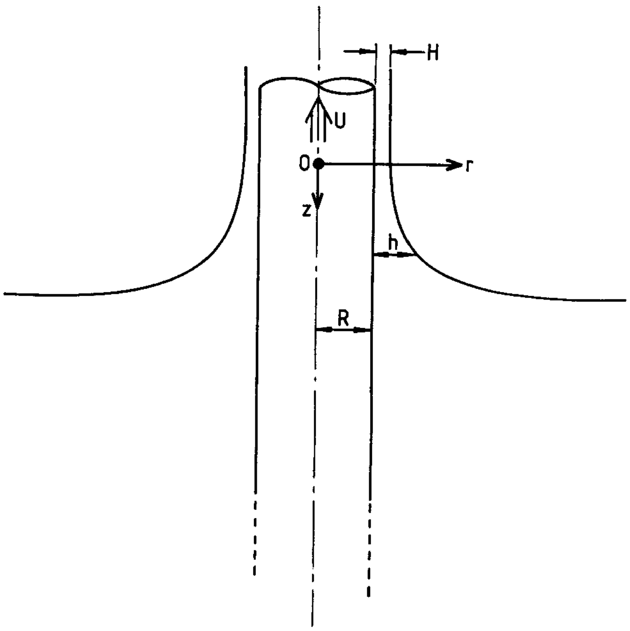

Figure 1. Coordinate system.

ducing two natural length scales:

$$d = (\mu U/\rho g)^{1/2}, \quad D = (\gamma/\rho g)^{1/2}. \tag{6}$$

Here d is the apparent scale of the film thickness h and D is the scale of the hydrostatic meniscus. Note that $d/D = (Ca)^{1/2}$.

Suppose now that $R \gg D$. Then we can expect the flat–plate solution to be a reasonable first approximation; this analysis shows that the correct scale for h is not d, but δd, where $\delta = (Ca)^{1/6}$. [This result was obtained first by Landau and Levich (1942).] We also find that the correct scale for z is $\delta d^{1/3} D^{2/3} = d/\delta$. Thus, we introduce a dimensionless film thickness φ and a dimensionless z cordinate ζ given by

$$h = \delta d \varphi, \quad z = \delta^{-1} d \zeta, \quad H = \delta d J. \tag{7}$$

Using these, and Eqs. 2 and 3 for the curvature κ, Eq. 5 becomes

$$\frac{1}{3}\varphi^3 \frac{d}{d\zeta}\left\{\frac{\varphi''}{(1+\delta^4\varphi'^2)^{3/2}} - \frac{\lambda}{(1+\delta^4\varphi'^2)^{1/2}(1+\delta^4\lambda\varphi)}\right\} = (\varphi - J) - \frac{1}{3}\delta^2(\varphi^3 - J^3). \tag{8}$$

In this equation $\lambda = D/R$ which we are supposing to be a small quantity. Recall also that we assume throughout that Ca is small and therefore δ is small. Some of the displayed terms in Eq. 8 are therefore negligible; but they are retained because they nontheless play a role in the matching to the meniscus region. To see this we change the variables to the ones which are appropriate in the meniscus, where both z and h will have length scale D. The correct dimensionless variables, Z and Φ say, are given by

$$z = DZ, \quad H = D\Phi \tag{9}$$

and Eq. 5 becomes, when certain small terms are omitted,

$$\frac{1}{3}\Phi^3 + \frac{1}{3}\Phi^3 \frac{d}{dZ} \cdot \left\{\frac{\Phi''}{(1+\Phi'^2)^{3/2}} - \frac{\lambda}{(1+\Phi'2)^{1/2}(1+\lambda\Phi)}\right\} = 0. \tag{10}$$

When the factor $\frac{1}{3}\Phi^3$ is removed, this can be seen to be the equation of an axisymmetric hydrostatic meniscus that can be integrated once to give:

$$\frac{\Phi''}{(1+\Phi'^2)^{3/2}} - \frac{\lambda}{(1+\Phi'^2)(1+\lambda\Phi)} = -Z + \text{constant}. \tag{11}$$

The technical details of the matching of the solutions of Eqs. 8 and 11 will not be given here since they are almost the same as the flat-plate case. It turns out that the solution of Eq. 11 that we need must correspond to a meniscus making zero contact angle with the cylinder. We have free choice of the origin of Z and so we put it at the contact level. Thus Φ satisfies

$$\Phi = \Phi' = 0 \text{ at } Z = 0. \tag{12}$$

The constant of integration in Eq. 11 can be identified as Z_m, the height of the contact circle above the horizontal free surface in the liquid bath. To see this, observe that the lefthand side of Eq. 11 is the curvature, which must vanish when $Z = Z_m$. What we need for the solution of Eq. 5 via the matching process is the value of Φ'' at the contact circle, that is $\Phi''(0)$, and using Eq. 12 we obtain

$$\Phi''(0) = \lambda + Z_m. \tag{13}$$

In this equation Z_m is an unknown function of λ which is determined from the solution of Eq. 11. When $\lambda = 0$ the flat plate case is recovered and $Z_m(0) = 2^{1/2}$. In general no closed form solution is available; there are various approximate and numerical solutions which will be discussed later.

The matching condition to be applied to the solution of Eq. 8 for φ is that $\varphi''(\zeta)$ must approach the value $\Phi''(0)$ as $\zeta \rightarrow \infty$ (when the appropriate scale changes are made).

Inspection of Eq. 8 now shows that when the small terms in δ are omitted (as they should be as a first approximation) the term in λ also disappears since it is a constant. This term corresponds to the pressure produced by the axial curvature of the free surface and since it is constant to this order of approximation there is no dynamical effect. Equation 8 reduces to

$$\varphi^3\varphi'' = 3(\varphi - J) \tag{14}$$

which is equivalent to Eq. 22 of Wilson (1982). The analysis in that paper, therefore, goes through with only the trivial change that the righthand side of Eq. 13 is carried as an implicit function of λ instead of the explicit value $2^{1/2}$. The solution can be taken to the next order of approximation in δ, and there results

$$h = \delta d \varphi = \delta d \left\{\frac{1.338}{Z_m + \lambda} - \frac{0.3022\,\delta^2}{(Z_m + \lambda)^3} \cdots\right\} \tag{15}$$

Next we observe that the assumption that λ is small has not actually been used; Eq. 13 is formally exact and λ disappeared from Eq. 8 because a constant differentiates to zero rather than because the term in λ is formally small. Thus, Eq. 15 holds *even when* λ *is of order unity*, provided (of course) that Z_m can be regarded as known.

It is not possible to extend this to the small wire case, for which λ is large, in this manner because of the evident clash in Eq. 8 between the smallness of δ and the largeness of λ. For this case the basic Eq. 5 must be reconsidered and new variables found which are appropriate to the case $R \ll D$. After some trial and error, it is found that the correct dimensionless film thickness and vertical coordinate, $\tilde{\varphi}$ and $\tilde{\zeta}$ say, are given by

$$h = R\delta^4\tilde{\varphi}, \quad z = R\delta^2\tilde{\zeta}, \quad H = R\delta^4\tilde{J}, \tag{16}$$

which may be compared with Eq. 7. The equation becomes

$$\frac{1}{3}\tilde{\varphi}^3\frac{d}{d\tilde{\zeta}}\left\{\frac{\tilde{\varphi}''}{(1+\delta^4\tilde{\varphi}'^2)^{3/2}} - \frac{1}{(1+\delta^4\tilde{\varphi})\,(1+\delta^4\tilde{\varphi}'^2)^{1/2}}\right\} = \tilde{\varphi} - \tilde{J} - \frac{\delta^2}{3\lambda^2}(\tilde{\varphi}^3 - \tilde{J}^3), \tag{17}$$

which corresponds to Eq. 8; note that here λ is supposed to be large.

As before the solution has to be matched to a hydrostatic meniscus, in which both h and z should be scaled on R. This scaling is not at all obvious; it follows from the work of Lo (1983), who showed that the meniscus has a two-region structure and that the proposed scales are correct near the cylinder. If, accordingly, we introduce new variables $\tilde{\Phi}$ and $\tilde{Z}$ given by

$$h = R\tilde{\Phi}, \quad z = R\tilde{Z}$$

we obtain

$$\frac{d}{d\tilde{Z}}\left\{\frac{\tilde{\Phi}''}{(1+\tilde{\Phi}'^2)^{3/2}} - \frac{1}{(1+\tilde{\Phi})(1+\tilde{\Phi}^2)^{1/2}}\right\} = -\frac{1}{\lambda^2} + 0(\delta^6) \tag{19}$$

and certain small terms have been omitted. As indicated these terms are of order δ^6, i.e., of order Ca; however, the term in λ^{-2} is also small and to proceed we shall have to postulate that $Ca \ll \lambda^{-2}$. The significance of this will be discussed later.

The analysis now proceeds much as in the previous case. We can integrate Eq. 19 and identify the constant as $\tilde{Z}_m$, so that

$$\frac{\tilde{\Phi}''}{(1+\tilde{\Phi}'^2)^{3/2}} - \frac{1}{(1+\tilde{\Phi})(1+\tilde{\Phi}'^2)^{1/2}} = -\frac{1}{\lambda^2}(\tilde{Z} - \tilde{Z}_m). \tag{20}$$

and what is needed is $\tilde{\Phi}''(0)$, which is given by

$$\tilde{\Phi}''(0) = 1 + \tilde{Z}_m/\lambda^2. \tag{21}$$

Corresponding to Eq. 15 we obtain

$$h = R\delta^4\left\{\frac{1.338}{1+\tilde{Z}_m/\lambda^2} - \frac{0.3022\,\delta^2}{\lambda^2(1+\tilde{Z}_m/\lambda^2)^3}\cdots\right\} \tag{22}$$

This expression is identical with Eq. 15 as can be seen by unscrambling the various changes of scale. The difference lies in the order of magnitude of the error, which depends on the relative magnitudes of δ and λ and is not of much significance. Either equation can be used for any value of λ provided it is not so large that λ^{-2} is comparable with Ca. When that happens the scheme fails, and the reason is that the meniscus is so small that viscous forces are significant everywhere in it and there is no genuine hydrostatic balance. (It should be stressed that no difficulty would be encountered in the evaluation of Eq. 15 or 22 for any value of λ; it is just that for very large λ the result will not be reliable because terms have been neglected which are of the same order as those retained.)

It remains to consider the evaluation of the meniscus height Z_m. As noted above, there is a numerical solution (Hildebrand et al., 1970), which can be fitted to within a few percent by an expression of the form

$$Z_m = \frac{2^{1/2}}{1 + 0.44\lambda^{0.88}}. \tag{23}$$

This is not very good when λ is very large; in that case the accurate asymptotic formula of Lo (1983) can be used.

Concluding Remarks

An approximate expression for the film thickness entrained has been found. It is valid when $Ca \ll 1$ and $Ca \ll \lambda^{-2}$. It must be emphasized that if these restrictions are not met then the equations do not permit any rational simplification at all. In particular, if Ca is of order unity, the lubrication Eq. 1 fails. This has not been realised by previous authors.

On the other hand there is not much difference numerically between the present results and previous theories (White and Tallmadge, 1967) and so no improved agreement with experiment can be expected.

The present formula does have the advantage that it is easy to use and is correct: i.e., it is a valid asymptotic approximation.

Notation

- r, z = cylindrical polar coordinates
- R = cylinder radius
- h = film thickness
- $s = R + h$
- H = film thickness far above the meniscus
- w = fluid velocity in vertical direction
- Q = total flux withdrawn
- p = fluid pressure
- g = acceleration due to gravity
- d = apparent scale of film thickness, $(\mu U/pg)^{1/2}$
- D = scale of hydrostatic meniscus, $(\gamma/pg)^{1/2}$
- Ca = capillary number, $\mu U/\gamma$.
- U = speed of withdrawal
- $\zeta, \tilde{\zeta}, Z, \tilde{Z}$ = dimensionless coordinates defined in Eqs. 7, 9, 16 and 18
- $\varphi, \Phi, \tilde{\varphi}, \tilde{\Phi}$ = dimensionless film thickness ditto
- $J, \tilde{J}$ = dimensionless version of H ditto
- Z_m = height reached by static meniscus (dimensionless)

Greek Letters

ρ = fluid density
μ = fluid viscosity
ν = fluid kinematic viscosity
$\delta = (Ca)^{1/6}$
$\lambda = D/R$
γ = surface tension
κ = total curvature of free surface
κ_R = radial curvature
κ_A = axial curvature

Literature Cited

Hildebrand, R. E., M. A. Hildebrand, and J. A. Tallmadge, "Effect of Radius on Static Menisci Heights on Wires," *Chem. Eng. J.*, **2**, 297 (1971).

Landau, L., and B. Levich, "Dragging of a Liquid by a Moving Plate," *Acta Physicochim. URSS*, **17**, 42 (1942).

Lo, L. L., "The Meniscus on a Needle—a Lesson in Matching," *J. Fluid Mech.*, **132**, 65 (1983).

Ruschak, K. J., "Boundary Conditions at a Liquid/Air Interface in Lubrication Flows," *J. Fluid Mech.*, **119**, 107 (1982).

White, D. A., and J. A. Tallmadge, "A Gravity Corrected Theory for Cylinder Withdrawal," *AIChE J.*, **13**, 745 (1967).

Williamson, A. S., "The Tearing of an Adhesive Layer between Flexible Tapes Pulled Apart," *J. Fluid Mech.*, **52**, 639 (1972).

Wilson, S. D. R., "The Drag-Out Problem in Film Coating Theory," *J. Eng. Math.*, **16**, 209 (1982).

Manuscript received Feb. 15, 1988, and revision received May 12, 1988.